

Quantifying Non-Markovianity with Temporal Steering

Shin-Liang Chen,¹ Neill Lambert,² Che-Ming Li,³ Adam Miranowicz,^{2,4} Yueh-Nan Chen,^{1,2,*} and Franco Nori^{2,5}

¹*Department of Physics and National Center for Theoretical Sciences, National Cheng-Kung University, Tainan 701, Taiwan*

²*CEMS, RIKEN, 351-0198 Wako-shi, Japan*

³*Department of Engineering Science and Supercomputing Research Center, National Cheng-Kung University, Tainan City 701, Taiwan*

⁴*Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*

⁵*Department of Physics, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA*

Einstein-Podolsky-Rosen (EPR) steering is a type of quantum correlation which allows one to remotely prepare, or steer, the state of a distant quantum system. While EPR steering can be thought of as a purely spatial correlation there does exist a temporal analogue, in the form of single-system temporal steering. However, a precise quantification of such temporal steering has been lacking. Here we show that it can be measured, via semidefinite programming, with a *temporal steerable weight*, in direct analogy to the recently proposed EPR steerable weight. We find a useful property of the temporal steerable weight in that it is a non-increasing function under completely-positive trace-preserving maps and can be used to define a sufficient and practical measure of strong non-Markovianity.

PACS numbers: 03.65.Ta, 03.67.Mn, 03.67.Bg

Quantum entanglement, Einstein-Podolsky-Rosen (EPR) steering, and Bell non-locality are three of the most intriguing phenomena in quantum physics and, in varying degrees, are thought to act as resources; fuel that powers a range of quantum technologies. Entanglement [1–3] comes in hand-in-hand with the complexity of quantum systems, and may be behind the potential speed-up of quantum computation. Bell non-locality and EPR steering are thought to be the driving power of quantum cryptography, and have both been recast in that language. For example, in a quantum key distribution scenario, two parties wish to generate a secret key using shared quantum states as a resource. If one party (Bob) trusts his own experimental apparatus but not that of the other party (Alice), a violation of a steering inequality [2–4] can be used to certify that true quantum correlations exist between their shared states. In stricter terms, such a test proves to Bob that the correlations he observes between his measurement results and Alice’s cannot be described by a local hidden state model; his state is truly being influenced by Alice’s measurements in a non-local manner. As with entanglement, one quantifies the amount of steering that is possible with a given shared state via a range of possible measures [5–8]. Very recently, a powerful example of such a measure, the steerable weight, was proposed by Skrzypczyk *et al.* [9, 10].

In EPR steering the notion of non-locality, via space-like separations between parties, plays an important role. If we relax this constraint, and consider time-like separation of measurement events, can the concept of steering still be used in a meaningful way? We can find inspiration in the fact that there do already exist other types of non-trivial temporal quantum correlations complementary to both Bell non-locality and entanglement. For the former, one of the most well-known examples is the Leggett-Garg (LG) inequality [11], which can be used to test the assumption of “macroscopic realism”, in contrast to the non-local realism tested by Bell’s Inequality, and for which experimental violations have been observed in a large range of systems [12–14]. For the latter, motivated by

the Choi-Jamiołkowski (CJ) isomorphism [15], which equates the correlations in a bi-partite quantum system with two-time correlations of a single quantum system, the notion of temporal entanglement has been proposed in various forms [16–22]. Returning to steering, the concept of temporal steering, and a temporal steering inequality, was recently introduced by Chen *et al.* [23]. Also inspired by the CJ isomorphism, they showed that, even without the assumption of non-locality, the concept of one party not trusting the earlier measurements made by another party delineates between certain classical and quantum correlations. Not only does this have direct practical applications in verifying a quantum channel for quantum key distribution (QKD), it was recently shown that temporal steering, like EPR steering [24, 25], is intimately linked to the concepts of realism and joint measurability [26–29].

Still lacking however is a measure to quantify these “temporal steering” quantum correlations. Here, in analogy to the EPR steerable weight [9, 10], we define the temporal-steerable-weight (temporal-SW) as a measure of temporal steering. We prove that the temporal-SW is non-increasing under a completely-positive trace-preserving (CPT) map and can be used to define a sufficient but not necessary measure of non-Markovianity. In the same way that the spatial steerable weight can be considered a measure of strong entanglement, since not every entangled state is steerable, we define the *temporal-SW* as a measure of *strong* non-Markovianity because it vanishes for weak non-Markovian process. (This is also in analogy to, e.g., the phenomenon of *strong non-classicality*, which can be detected and quantified by a weaker criterion of non-classicality [30, 31]). We show this by comparing the non-Markovianity measured by the temporal-SW to an existing entanglement-based measure [32], and find that it is, as expected, less-sensitive. However, the temporal-SW is, in principle, easier to implement experimentally, as it does not require the use of an ancilla, nor full process tomography. These results, together with a few illustrative examples discussed mainly in the Supplementary Material [33], suggest

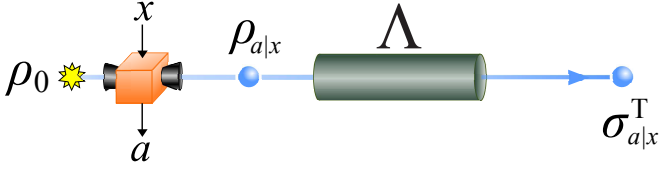


FIG. 1. (Color online) Schematic diagram of temporal steering. In the beginning, Alice performs the measurement $F_{a|x} = M_{a|x}^\dagger M_{a|x}$ on an initial state ρ_0 . Then, ρ_0 is mapped to $\rho_{a|x}$ and sent into a quantum channel Λ . Finally, Bob receives the assemblage $\{\sigma_{a|x}^T\}$ at time t .

that temporal steering can serve as a unique and useful quantum resource.

Temporal steerable weight.— Now we introduce the concept of a temporal steerable weight in analogy to the spatial steerable weight introduced recently by Skrzypczyk *et al.* [9, 10]. In the standard (spatial) EPR steering scenario, Alice performs a POVM (positive-operator valued measure) measurement $F_{a|x} = M_{a|x}^\dagger M_{a|x}$, $\sum_a F_{a|x} = \mathbb{1}$, on a state ρ_{AB} shared with Bob and creates the assemblage $\{\sigma_{a|x}\}$, where a is the measurement result and x is the basis of the measurement. In defining a steering inequality, or a steerable weight, one assumes that Bob does not trust Alice, nor her experimental apparatus, and wishes to distinguish between true manipulation of his local state via quantum correlations and correlations that cannot be distinguished from some classical theory, typically a local hidden state model. In temporal steering, we also let Alice perform a POVM measurement $F_{a|x} = M_{a|x}^\dagger M_{a|x}$ but on a single system in an initial state ρ_0 at time $t = 0$. After the measurement, the initial state is mapped to $\rho_{a|x}$ (see Fig. 1):

$$\rho_0 \mapsto \rho_{a|x} = \frac{M_{a|x} \rho_0 M_{a|x}^\dagger}{p(a|x)}, \quad (1)$$

with the probability $p(a|x) = \text{tr}(M_{a|x} \rho_0 M_{a|x}^\dagger)$. After this initial measurement, the state $\rho_{a|x}$ is sent into a quantum channel Λ for a time t . At time t , Bob receives the system and performs quantum tomography to obtain the state $\sigma_{a|x}$, i.e., $\Lambda(\rho_{a|x}) = \sigma_{a|x}$. To mimic the unnormalized assemblage [9, 10] in standard EPR-steering, we define the unnormalized states in temporal steering

$$\sigma_{a|x}^T \equiv p(a|x) \sigma_{a|x}, \quad (2)$$

where the superscript T reminds one that the assemblage $\{\sigma_{a|x}^T\}$ is for temporal steering.

However, the quantum channel may be noisy, obliterating the influence of Alice’s measurement choice, or Alice’s measurement results could have been fabricated via classical strategies. In these cases, $\sigma_{a|x}^T$ may include, or be entirely described by, an unsteerable assemblage which we define as

$$\sigma_{a|x}^{\text{T,US}} = \sum_{\lambda} P(\lambda) P(a|x|\lambda) \sigma_{\lambda} \quad (3)$$

where $\sum_{\lambda} P(\lambda) = 1$. We have written the result a , conditional on the basis x , with a subscript notation $a_{|x} \equiv a|x$. In the EPR setting λ represents a local hidden variable which determines the possible correlations between Alice’s and Bob’s measurement results from a source which obeys classical realism. As in that case, when Alice reveals her measurement results, Bob can update his knowledge of his state, as indicated by two equal forms (by applying the chain rule), $\sum_{\lambda} P(\lambda) P(a_{|x}|\lambda) \sigma_{\lambda} = \sum_{\lambda} P(a_{|x}) P(\lambda|a_{|x}) \sigma_{\lambda}$. Then the unsteerable states are those states which obey the classical (realism) chain rule for Alice’s joint measurement results, as shown in a recent work on steering witnesses [26]. No matter what happens during the transmission, Bob’s task is to check whether the assemblage he receives can be written in the hidden-state form [Eq. (3)] or not. If he can, this means the state Bob receives is independent of the basis x Alice chooses to measure in. As mentioned above, this may be because the quantum channel is too noisy, such that the influence of Alice’s measurements is no longer discernable, or Alice’s measurement results could have been fabricated via classical strategies. On the other hand, if the assemblage Bob receives cannot be written in the form of Eq. (3), he is convinced that Alice has influenced his state by her choice of measurement. In this case, we call the assemblage Bob receives “temporally steerable” and is symbolized as $\{\sigma_{a|x}^{\text{T,S}}\}$.

To determine the steerable weight, one considers the overlap between the state Bob receives and the unsteerable assemblage, such that his state can be written as a mixture

$$\sigma_{a|x}^T = \mu \sigma_{a|x}^{\text{T,US}} + (1 - \mu) \sigma_{a|x}^{\text{T,S}}. \quad (4)$$

To quantify the “steerability in time” for a given assemblage $\{\sigma_{a|x}^T\}$, one has to maximize μ , i.e., maximize the proportion of $\sigma_{a|x}^{\text{T,US}}$. Then, the “temporal steerable weight” can be defined as $\text{TSW} = 1 - \mu^*$, in which μ^* is the maximum of μ and can be obtained from semidefinite programming [9, 10, 36]:

$$\begin{aligned} \mu^* &= \max \text{tr} \sum_{\lambda} \tilde{\sigma}_{\lambda} \\ \text{subject to } & \sigma_{a|x}^T - \sum_{\lambda} D_{\lambda}(a|x) \tilde{\sigma}_{\lambda} \geq 0 \quad \forall a, x \\ & \tilde{\sigma}_{\lambda} \geq 0 \quad \forall \lambda, \end{aligned} \quad (5)$$

where $\tilde{\sigma}_{\lambda} = \mu \sigma_{\lambda}$, and $D_{\lambda}(a|x)$ are the extremal deterministic values [9] of the conditional probability distributions $P(a_{|x}|\lambda)$. Equation (5), which is formulated as a semi-definite program (SDP), can be numerically implemented in various convex optimization packages, e.g., Refs. [37, 38].

So far, the formalism is parallel to the standard EPR steerable weight [9]. The primary difference is that $\{\sigma_{a|x}\}$ in Ref. [9] is created through the entanglement between Alice and Bob. Here, $\{\sigma_{a|x}^T\}$ is created through Alice’s measurement and the influence of the quantum channel Λ . In the Supplementary Material [33], we give an explicit pedagogical example of how to evaluate the temporal steerable weight.

Measure of non-Markovianity.— Now we apply the introduced temporal steering weight as a measure of non-

Markovianity. Non-Markovianity is a term used to describe the situation when an environment surrounding a quantum system has memory of its past evolution. It is an important concept both because many natural and man-made quantum systems exist in a regime where the assumption of a Markovian (memory-less) environment fails, but also because it can lead to counter-intuitive results regarding the decay of quantum effects, particularly when the quantum system is strongly coupled to the surrounding environment. There has been a range of efforts at constructing measures of non-Markovianity, typically based on a scenario where the time evolution of a quantum system is analysed for non-Markovian properties. Arguably, the most popular measures of non-Markovianity were introduced in Refs. [32, 39]. Recently, an attempt to classify these non-Markovianity measures in a unified framework was described in Ref. [40]. Useful for us here is the approach taken in [39], which is based on observing the behavior of the trace distance between two quantum states. They derived a measure of non-Markovianity by noting that all CPT maps Φ are contractions of the trace distance metric, and a given dynamic process is defined as Markovian if the map is divisible, i.e. $\Phi(\tau + t, 0) = \Phi(\tau + t, t)\Phi(t, 0)$, for all positive t and τ . These two properties lead to the monotonicity of the trace distance, and violations of this monotonicity indicate the occurrence of non-Markovian dynamics. In a similar way, below we prove that the temporal-SW of a system undergoing a CPT map is also a non-increasing function, i.e.,

$$\text{TSW}_\rho \geq \text{TSW}_{\Phi(\tau)\rho} \quad (6)$$

for a CPT map $\Phi(\tau)$. Together with the property of divisibility, one can conclude that the temporal-SW decreases monotonically under Markovian dynamics. Therefore our measure of non-Markovianity is defined by integrating the positive slope of the temporal-SW

$$\mathcal{N}_{\text{TSW}} \equiv \int_{\sigma_{\text{TSW}} > 0} dt \sigma_{\text{TSW}}(t, \rho_0, \Phi), \quad (7)$$

where $\sigma_{\text{TSW}}(t, \rho_0, \Phi) = \frac{d}{dt} \text{TSW}_{\Phi(t)\rho_0}$ is the rate of change of the temporal steerable weight. In the examples discussed in the Supplementary Material [33], we demonstrate explicitly how one can use this as a practical measure of strong non-Markovianity. Here we discuss only the following example.

Proof of the monotonicity of temporal-SW under Markovian dynamics.— First, we prove that the temporal-SW of a system undergoing a CPT map is a non-increasing function, as given by Eq. (6). To obtain the temporal-SW of a qubit at time t_1 , one needs the quantity, $\sigma_{a|x}^T(t_1) - \sum_{\lambda_1} D_{\lambda_1}(a|x) \tilde{\sigma}_{\lambda_1}$, in which the set $\{\tilde{\sigma}_{\lambda_1}\}$ is chosen to maximize $\text{Tr}(\sum_{\lambda_1} \tilde{\sigma}_{\lambda_1})$ at time t_1 . Summing all the measurement outcomes a and taking

the trace, we have

$$\begin{aligned} & \text{Tr} \left[\sum_a \sigma_{a|x}^T(t_1) - \sum_a \sum_{\lambda_1} D_{\lambda_1}(a|x) \tilde{\sigma}_{\lambda_1} \right] \\ &= \text{Tr} \left[\sum_a \sigma_{a|x}^T(t_1) - \sum_{\lambda_1} \tilde{\sigma}_{\lambda_1} \right] = 1 - \mu_1^*, \end{aligned} \quad (8)$$

where we have used the properties $\sum_a D_{\lambda}(a|x) = 1$, and $\text{Tr}[\sum_a \sigma_{a|x}^T(t_1)] = 1$. Similarly, to obtain the temporal-SW of the qubit at a later time $t_2 = t_1 + \tau$, one also has

$$\text{Tr} \left[\sum_a \sigma_{a|x}^T(t_2) - \sum_a \sum_{\lambda_2} D_{\lambda_2}(a|x) \tilde{\sigma}_{\lambda_2} \right] = 1 - \mu_2^*, \quad (9)$$

where $\{\tilde{\sigma}_{\lambda_2}\}$ is chosen to maximize $\text{Tr}(\sum_{\lambda_2} \tilde{\sigma}_{\lambda_2})$ at time t_2 . One can also perform a CPT map $\Phi(\tau)$ to Eq. (8), giving

$$\begin{aligned} & \text{Tr} \left[\sum_a \Phi(\tau) \sigma_{a|x}^T(t_1) - \sum_a \sum_{\lambda_1} \Phi(\tau) D_{\lambda_1}(a|x) \tilde{\sigma}_{\lambda_1} \right] \\ &= \text{Tr} \left[\sum_a \sigma_{a|x}^T(t_2) - \sum_a \sum_{\lambda_1} D_{\lambda_1}(a|x) (\Phi(\tau) \tilde{\sigma}_{\lambda_1}) \right] \end{aligned} \quad (10)$$

Since $\Phi(\tau)$ is a trace-preserving map, the value of Eq. (10) is still $1 - \mu_1^*$. However, we know that the set $\{\tilde{\sigma}_{\lambda_2}\}$ is the optimal way to maximize $\text{Tr}(\sum_{\lambda_2} \tilde{\sigma}_{\lambda_2})$ at time t_2 for Eq. (9). Therefore, comparing Eq. (9) with Eq. (10) would give

$$1 - \mu_1^* \geq 1 - \mu_2^*, \quad (11)$$

This proves the theorem given in Eq. (6). Employing the divisibility of Markovian dynamics leads to the monotonicity of the temporal-SW:

$$\begin{aligned} \text{TSW}\{\Phi(\tau + t, 0) \sigma_{a|x}\} &= \text{TSW}\{\Phi(\tau + t, t) \Phi(t, 0) \sigma_{a|x}\} \\ &\leq \text{TSW}\{\Phi(t, 0) \sigma_{a|x}\} \end{aligned} \quad (12)$$

An example of non-Markovianity of a spin-boson problem.— Exact solutions to the general spin-boson problem have applications in a huge range of systems, from quantum computing to physical chemistry and photosynthesis [41]. Various techniques and methods exist to numerically acquire such solutions, one of the most powerful of which is the hierarchy equations of motion [42, 43]. Here we use those equations to model a two-level system coupled to a bosonic environment or reservoir. The general Hamiltonian is written as

$$H_{\text{SB}} = \frac{E}{2} \sigma_z + \Delta \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{k}} \sigma_z \otimes l_{\mathbf{k}} (a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}), \quad (13)$$

where Δ is the two-level system tunneling amplitude, and E is the two-level system splitting. The environment modes are described by creation ($a_{\mathbf{k}}^\dagger$) and annihilation operators ($a_{\mathbf{k}}$) with energy $\omega_{\mathbf{k}}$, which couple to the system, described by the Pauli operators σ_z and σ_x , with strength $l_{\mathbf{k}}$. By assuming

that the environment modes are well-described by a Drude-Lorentz spectral density, $J(\omega) = 2\alpha\omega_c \frac{\omega}{\omega^2 + \omega_c^2}$, where α is the system-reservoir coupling strength and ω_c is the bath cut-off frequency, we can exactly solve the dynamics of the two-level system (details can be found in Ref. [41–43]). We can then compare the non-Markovianity as measured via the temporal-SW to that given by the non-monotonic behavior of the entanglement, as given by the concurrence [1], between the two-level system and an isolated ancilla [32]. One important difference in the two approaches is that in the temporal-SW case there is no ancilla. In the ancilla case, the initial condition between the system and ancilla is that of a maximally-entangled state; to mimic that in the temporal-SW case we assume the two-level system is initially in a maximally-mixed state. We then evolve the entire system-reservoir equations of motion, using parameters relevant to energy transfer in photosynthesis [41], and plot both measures in Fig. 2.

For both measures we see similar behavior, particularly as a function of reservoir cut-off frequency and reservoir temperature. However, as a function of system-reservoir coupling, the entanglement measure has a larger window of detection. This may be attributed to the hierarchical relationship between EPR steering and entanglement. For example, Ref. [2] has shown that EPR steerable states are a superset of Bell non-local states, and a subset of entangled states. This hierarchy links together these three different notions of quantum correlations. Therefore, the fact that the concurrence-based measure of non-Markovianity is more sensitive to the non-Markovianity than the temporal-SW measure seems linked, intuitively, to the notion that steering, in its EPR form, is a subset of entangled states. Also note that, the sharp features in both measures are typical, and arise because of the sudden vanishing and reappearance of both quantities in the temporal domain. Note that here, for consistency with Ref. [32], we plot \mathcal{N}_{TSW} and \mathcal{N}_C using

$$\mathcal{N}_i = \int_{t_0}^{t_f} \left| \frac{df_i[\rho(t)]}{dt} \right| dt + f_i[\rho(t_f)] - f_i[\rho(t_0)], \quad (14)$$

where for the temporal-SW measure $i = \text{TSW}$, the function $f_i[\rho(t)]$ is the temporal-SW at time t , while for the concurrence measure, $i = C$, the function $f_i[\rho(t)]$ is the concurrence between system and ancilla at time t . This definition for the integral differs from Eq. (7) by a trivial factor of 1/2.

Conclusions.— To summarize, we have discussed the concepts of “temporal” steering and how this can be quantified in a similar way to that of the original spatial EPR-steering. We further proved that the temporal steerable weight decreases monotonically under a CPT map and can be used as a measure of non-Markovianity, suggesting that both forms of steering can act as a quantum resource, similar to entanglement. Finally we note that, in parallel, the temporal steerable weight has been recently implemented experimentally [44].

Acknowledgement.— The concept of a temporal steerable weight was developed here independently of the parallel work [44]. S.-L. C. thanks H.-B. Chen for the discussion

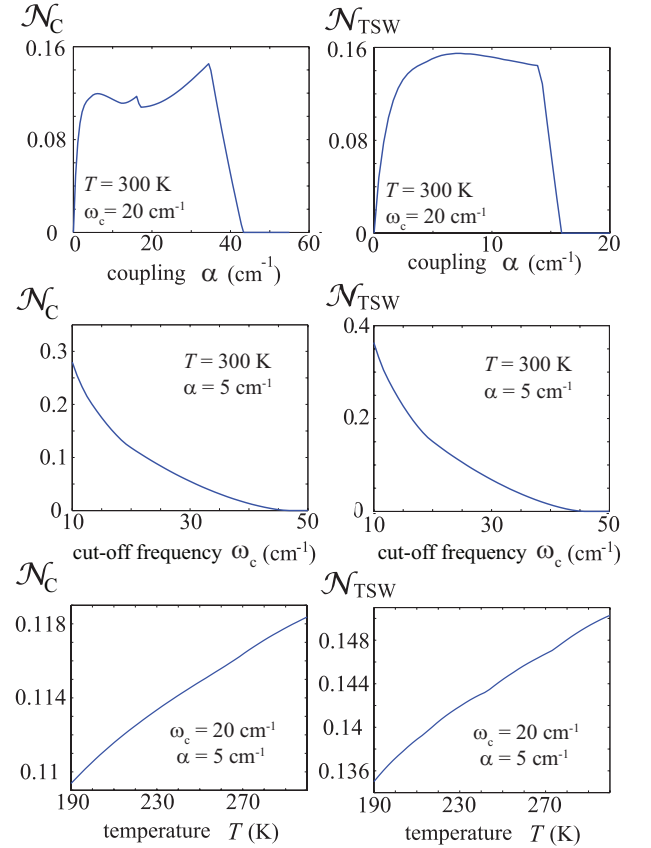


FIG. 2. (Color online) The non-Markovianity measures, \mathcal{N}_{TSW} (based on the temporal steerable weight) and \mathcal{N}_C (based on the entanglement with an ancilla), as a function of system-reservoir coupling α , reservoir cut-off frequency ω_c and temperature T , for a two-level system coupled to a bosonic reservoir with Drude-Lorentz spectral density. The system parameters are chosen to maximize the affect of the reservoir memory, with $E = 0$ and $\Delta = 100\text{cm}^{-1}$. The magnitude of these parameters are typical for excitation energy transfer in photosynthesis [41], where the memory effect and structure of the environment is thought to play an important role.

on quantum non-Markovianity. This work is supported partially by the National Center for Theoretical Sciences and Ministry of Science and Technology, Taiwan, grant number MOST 103-2112-M-006-017-MY4. A. M. is supported by the Polish National Science Centre under Grants DEC-2011/03/B/ST2/01903 and DEC-2011/02/A/ST2/00305. A. M. gratefully acknowledges a long-term fellowship from the Japan Society for the Promotion of Science (JSPS). F. N. is partially supported by the RIKEN iTHES Project, the MURI Center for Dynamic Magneto-Optics via the AFOSR award number FA9550-14-1-0040, the IMPACT program of JST, and a Grant-in-Aid for Scientific Research (A).

* yuehnan@mail.ncku.edu.tw

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki,

- “Quantum entanglement,” *Rev. Mod. Phys.* **81**, 865 (2009).
- [2] H. M. Wiseman, S. J. Jones, and A. C. Doherty, “Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox,” *Phys. Rev. Lett.* **98**, 140402 (2007).
- [3] S. J. Jones, H. M. Wiseman, and A. C. Doherty, “Entanglement, einstein-podolsky-rosen correlations, bell nonlocality, and steering,” *Phys. Rev. A* **76**, 052116 (2007).
- [4] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, “Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox,” *Phys. Rev. A* **80**, 032112 (2009).
- [5] S. Jevtic, M. Pusey, D. Jennings, and T. Rudolph, “Quantum steering ellipsoids,” *Phys. Rev. Lett.* **113**, 020402 (2014).
- [6] I. Kogias, A. R. Lee, S. Ragy, and G. Adesso, “Quantification of Gaussian quantum steering,” *Phys. Rev. Lett.* **114**, 060403 (2015).
- [7] Q. Y. He, Q. H. Gong, and M. D. Reid, “Classifying directional gaussian entanglement, einstein-podolsky-rosen steering, and discord,” *Phys. Rev. Lett.* **114**, 060402 (2015).
- [8] M. Piani and J. Watrous, “Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering,” *Phys. Rev. Lett.* **114**, 060404 (2015).
- [9] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, “Quantifying Einstein-Podolsky-Rosen steering,” *Phys. Rev. Lett.* **112**, 180404 (2014).
- [10] M. F. Pusey, “Negativity and steering: A stronger Peres conjecture,” *Phys. Rev. A* **88**, 032313 (2013).
- [11] A. J. Leggett and A. Garg, “Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks?” *Phys. Rev. Lett.* **54**, 857 (1985).
- [12] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, “Experimental violation of a Bell’s inequality in time with weak measurement,” *Nat. Phys.* **6**, 442 (2010).
- [13] G. C. Knee, S. Simmons, E. M. Gauger, J. J. L. Morton, H. Riemann, N. V. Abrosimov, P. Becker, H. J. Pohl, K. M. Itoh, M. L. W. Thewalt, G. A. D. Briggs, and S. C. Benjamin, “Violation of a Leggett-Garg inequality with ideal non-invasive measurements,” *Nat. Commun.* **3**, 606 (2012).
- [14] C. Emary, N. Lambert, and F. Nori, “Leggett-Garg inequalities,” *Rep. Prog. Phys.* **77**, 016001 (2014).
- [15] A. Jamiołkowski, “Linear transformations which preserve trace and positive semidefiniteness of operators,” *Rep. Math. Phys.* **3**, 275 (1972).
- [16] Č. Brukner, S. Taylor, S. Cheung, and V. Vedral, “Quantum Entanglement in Time,” arXiv:quant-ph/0402127 (2004).
- [17] T. Fritz, “Quantum correlations in the temporal Clauser-Horne-Shimony-Holt (CHSH) scenario,” *New J. Phys.* **12**, 083055 (2010).
- [18] S. J. Olson and T. C. Ralph, “Entanglement between the future and the past in the quantum vacuum,” *Phys. Rev. Lett.* **106**, 110404 (2011).
- [19] S. J. Olson and T. C. Ralph, “Extraction of timelike entanglement from the quantum vacuum,” *Phys. Rev. A* **85**, 012306 (2012).
- [20] C. Sabín, B. Peropadre, M. del Rey, and E. Martín-Martínez, “Extracting past-future vacuum correlations using circuit qed,” *Phys. Rev. Lett.* **109**, 033602 (2012).
- [21] E. Megidish, A. Halevy, T. Shacham, T. Dvir, L. Dovrat, and H. S. Eisenberg, “Entanglement swapping between photons that have never coexisted,” *Phys. Rev. Lett.* **110**, 210403 (2013).
- [22] J. Fitzsimons, J. Jones, and V. Vedral, “Quantum correlations which imply causation,” arXiv:1302.2731 (2013).
- [23] Y.-N. Chen, C.-M. Li, N. Lambert, S.-L. Chen, Y. Ota, G.-Y. Chen, and F. Nori, “Temporal steering inequality,” *Phys. Rev. A* **89**, 032112 (2014).
- [24] M. T. Quintino, T. Vértesi, and N. Brunner, “Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality,” *Phys. Rev. Lett.* **113**, 160402 (2014).
- [25] R. Uola, T. Moroder, and O. Gühne, “Joint measurability of generalized measurements implies classicality,” *Phys. Rev. Lett.* **113**, 160403 (2014).
- [26] C.-M. Li, Y.-N. Chen, N. Lambert, C.-Y. Chiu, and F. Nori, “Certifying single-system steering for quantum-information processing,” *Phys. Rev. A* **92**, 062310 (2015).
- [27] M. F. Pusey, “Verifying the quantumness of a channel with an untrusted device,” *J. Opt. Soc. Am. B* **32**, A56 (2015).
- [28] M. Piani, “Channel steering,” *J. Opt. Soc. Am. B* **32**, A1 (2015).
- [29] H. S. Karthik, J. Prabhu Tej, A. R. Usha Devi, and A. K. Rajagopal, “Joint measurability and temporal steering,” *J. Opt. Soc. Am. B* **32**, A34 (2015).
- [30] V. V. Dodonov and V. I. Man’ko, eds., *Theory of Nonclassical States of Light* (Taylor & Francis, London, 2002).
- [31] Arvind, N. Mukunda, and R. Simon, “Gaussian-wigner distributions and hierarchies of nonclassical states in quantum optics: The single-mode case,” *Phys. Rev. A* **56**, 5042 (1997).
- [32] A. Rivas, S. F. Huelga, and M. B. Plenio, “Entanglement and non-Markovianity of quantum evolutions,” *Phys. Rev. Lett.* **105**, 050403 (2010).
- [33] For a few illustrative examples, see Supplemental Material [url], which includes Refs. [34, 35].
- [34] E.-M. Laine, J. Piilo, and H.-P. Breuer, “Measure for the non-Markovianity of quantum processes,” *Phys. Rev. A* **81**, 062115 (2010).
- [35] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, 2004).
- [36] L. Vandenberghe and S. Boyd, “Semidefinite programming,” *SIAM Review* **38**, 49 (1996).
- [37] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming,” (2008).
- [38] M. Andersen, J. Dahl, and K. Vandenberghe, *CVXOPT: Python software for convex optimization* (2014). url:<http://cvxopt.org/>.
- [39] H.-P. Breuer, E.-M. Laine, and J. Piilo, “Measure for the degree of non-Markovian behavior of quantum processes in open systems,” *Phys. Rev. Lett.* **103**, 210401 (2009).
- [40] D. Chruściński and S. Maniscalco, “Degree of non-markovianity of quantum evolution,” *Phys. Rev. Lett.* **112**, 120404 (2014).
- [41] A. Ishizaki and G. R. Fleming, “Theoretical examination of quantum coherence in a photosynthetic system at physiological temperature,” *J. Chem. Phys.* **130**, 234111 (2009).
- [42] Y. Tanimura, “Nonperturbative expansion method for a quantum system coupled to a harmonic-oscillator bath,” *Phys. Rev. A* **41**, 6676 (1990).
- [43] T. Yoshitaka and K. Ryogo, “Time evolution of a quantum system in contact with a nearly Gaussian-Markoffian noise bath,” *J. Phys. Soc. Japan* **58**, 101 (1989).
- [44] K. Bartkiewicz, A. Černoch, K. Lemr, A. Miranowicz, and F. Nori, “Experimental temporal steering and security of quantum key distribution with mutually-unbiased bases,” arxiv:1503.00612 (2015).

Quantifying Non-Markovianity with Temporal Steering: Supplementary Material

Shin-Liang Chen,¹ Neill Lambert,² Che-Ming Li,³ Adam Miranowicz,^{2,4} Yueh-Nan Chen,^{1,2,*} and Franco Nori^{2,5}

¹Department of Physics and National Center for Theoretical Sciences, National Cheng-Kung University, Tainan 701, Taiwan

²CEMS, RIKEN, 351-0198 Wako-shi, Japan

³Department of Engineering Science and Supercomputing Research Center, National Cheng-Kung University, Tainan City 701, Taiwan

⁴Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland

⁵Department of Physics, The University of Michigan, Ann Arbor, MI 48109-1040, USA

In this supplementary material we give a few illustrative examples of the calculation of the temporal steerable weight and its application as a measure of strong non-Markovianity for some prototype models.

HOW TO CALCULATE THE STEERABLE WEIGHT: A PEDAGOGICAL EXAMPLE

Here we show explicitly how to calculate the steerable weight of Skrzypczyk *et al.* [1] in a simple example. Specifically, we assume three types of measurements corresponding to the projections on the eigenstates of the Pauli operators:

$$\begin{aligned} X &= |+\rangle\langle+| - |-\rangle\langle-|, \\ Y &= |R\rangle\langle R| - |L\rangle\langle L|, \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1|, \end{aligned} \quad (1)$$

where $|0\rangle = |H\rangle$, $|1\rangle = |V\rangle$, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, $|R\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$, and $|L\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$, which can be interpreted as: horizontal, vertical, diagonal, anti-diagonal, right-circular, and left-circular polarization states for the optical polarization qubits, respectively. We can label the eigenstates of the Pauli operators together with their eigenvalues as follows: $|x_1\rangle = |+\rangle$ with $x_1 = +1$, $|x_2\rangle = |-\rangle$ with $x_2 = -1$, $|y_1\rangle = |R\rangle$ with $y_1 = +1$, ..., and $|z_2\rangle = |1\rangle$ with $z_2 = -1$.

Then, possible unnormalized states of Bob $\sigma_{a|x}$ ($x = X, Y, Z$) for a given two-qubit state ρ read

$$\begin{aligned} \sigma_{a|x}^{(1)} &\equiv \sigma_{+1|X} = \text{Tr}_A[|+\rangle\langle+| \otimes I] \rho, \\ \sigma_{a|x}^{(2)} &\equiv \sigma_{-1|X} = \text{Tr}_A[|-\rangle\langle-| \otimes I] \rho, \\ \sigma_{a|x}^{(3)} &\equiv \sigma_{+1|Y} = \text{Tr}_A[|R\rangle\langle R| \otimes I] \rho, \\ \sigma_{a|x}^{(4)} &\equiv \sigma_{-1|Y} = \text{Tr}_A[|L\rangle\langle L| \otimes I] \rho, \\ \sigma_{a|x}^{(5)} &\equiv \sigma_{+1|Z} = \text{Tr}_A[|0\rangle\langle 0| \otimes I] \rho, \\ \sigma_{a|x}^{(6)} &\equiv \sigma_{-1|Z} = \text{Tr}_A[|1\rangle\langle 1| \otimes I] \rho, \end{aligned} \quad (2)$$

where I is the single-qubit identity operator. A classical random variable held by Alice,

$$\lambda_n = [x_i, y_j, z_k] \equiv [\langle x_i|X|x_i\rangle, \langle y_j|Y|y_j\rangle, \langle z_k|Z|z_k\rangle], \quad (3)$$

can take the following values:

$$\begin{aligned} \lambda_1 &= [-1, -1, -1], & \lambda_2 &= [-1, -1, +1], \\ \lambda_3 &= [-1, +1, -1], & \lambda_4 &= [-1, +1, +1], \\ \lambda_5 &= [+1, -1, -1], & \lambda_6 &= [+1, -1, +1], \\ \lambda_7 &= [+1, +1, -1], & \lambda_8 &= [+1, +1, +1]. \end{aligned} \quad (4)$$

The extremal deterministic single-party conditional probability distributions for Alice read

$$\begin{aligned} [D_{\lambda_1}(+1|X), \dots, D_{\lambda_8}(+1|X)] &= [0, 0, 0, 0, 1, 1, 1, 1], \\ [D_{\lambda_1}(-1|X), \dots, D_{\lambda_8}(-1|X)] &= [1, 1, 1, 1, 0, 0, 0, 0], \\ &\vdots \\ [D_{\lambda_1}(-1|Z), \dots, D_{\lambda_8}(-1|Z)] &= [1, 0, 1, 0, 1, 0, 1, 0]. \end{aligned} \quad (5)$$

Let us denote an unsteerable assemblage as

$$\sigma_{a|x}^{\text{US}} \equiv \sum_{\lambda} D_{\lambda}(a|x) \sigma_{\lambda} = \sum_{n=1}^8 D_{\lambda_n}(a|x) \sigma_{\lambda_n}. \quad (6)$$

Then, we have

$$\begin{aligned} \sigma_{a|x}^{(1)\text{US}} &\equiv \sigma_{+1|X}^{\text{US}} = \sigma_{\lambda_5} + \sigma_{\lambda_6} + \sigma_{\lambda_7} + \sigma_{\lambda_8}, \\ \sigma_{a|x}^{(2)\text{US}} &\equiv \sigma_{-1|X}^{\text{US}} = \sigma_{\lambda_1} + \sigma_{\lambda_2} + \sigma_{\lambda_3} + \sigma_{\lambda_4}, \\ \sigma_{a|x}^{(3)\text{US}} &\equiv \sigma_{+1|Y}^{\text{US}} = \sigma_{\lambda_3} + \sigma_{\lambda_4} + \sigma_{\lambda_7} + \sigma_{\lambda_8}, \\ \sigma_{a|x}^{(4)\text{US}} &\equiv \sigma_{-1|Y}^{\text{US}} = \sigma_{\lambda_1} + \sigma_{\lambda_2} + \sigma_{\lambda_5} + \sigma_{\lambda_6}, \\ \sigma_{a|x}^{(5)\text{US}} &\equiv \sigma_{+1|Z}^{\text{US}} = \sigma_{\lambda_2} + \sigma_{\lambda_4} + \sigma_{\lambda_6} + \sigma_{\lambda_8}, \\ \sigma_{a|x}^{(6)\text{US}} &\equiv \sigma_{-1|Z}^{\text{US}} = \sigma_{\lambda_1} + \sigma_{\lambda_3} + \sigma_{\lambda_5} + \sigma_{\lambda_7}. \end{aligned} \quad (7)$$

The steerable weight SW can be given as the solution of the following semidefinite program: Find

$$\text{SW} = 1 - \max \text{Tr} \left(\sum_{n=1}^8 \sigma_{\lambda_n} \right) \quad (8)$$

such that

$$\left(\sigma_{a|x}^{(i)} - \sigma_{a|x}^{(i)\text{US}} \right) \geq 0 \quad \text{and} \quad \sigma_{\lambda_n} \geq 0 \quad (9)$$

for $i = 1, 2, \dots, 6$ and $n = 1, \dots, 8$. By using a numerical package for convex optimization [2–4], one can implement this semidefinite program in a straightforward way. This is easily generalized to the temporal case by replacing the two-qubit measurements in Eq. (2) with measurements on a single qubit, followed by evolution under the channel Λ .

EXAMPLE 1: COHERENT RABI OSCILLATIONS OF A MARKOVIAN SYSTEM

As a first simple example of the behavior of the temporal-SW under a Markovian dynamics, we consider a qubit that

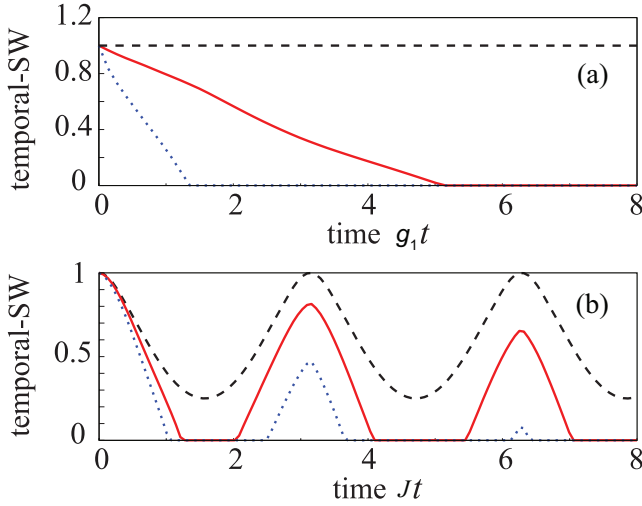


FIG. 1. (Color online) The temporal steerable weight (temporal-SW) as a function of evolution time when a system is in (a) a Markovian environment (example 1) and (b) non-Markovian environment (example 2). (a) The temporal-SW when the system undergoes coherent Rabi oscillations and purely Markovian decay (example 1). The black dashed, red solid, and blue dotted curves represent the results of the decay rate $\gamma_1/g_1 = 0, 1/6$, and 1 , respectively. The time t is in units of $1/g_1$, and \hbar is set to 1 . (b) The temporal-SW when the system interacts with a non-Markovian environment (example 2). The black dashed, red solid, and blue dotted curves represent the results of the decay rate $\gamma_2/J = 0, 0.03$, and 0.1 , respectively. Here, the time t is in units of $1/J$.

undergoes coherent Rabi oscillations and purely Markovian decay. The Hamiltonian of the system is

$$H = \hbar g_1 (\sigma_+ + \sigma_-), \quad (10)$$

where $\hbar g_1$ is the coherent coupling strength between two eigenstates, $|+\rangle$ and $|-\rangle$, of the qubit, and $\sigma_+ = |+\rangle\langle-|$ and $\sigma_- = |-\rangle\langle+|$ can be considered the raising and lowering operators, respectively. A Markovian channel induces a dissipation rate γ_1 from $|+\rangle$ to $|-\rangle$. We assume that the initial state, ρ_0 in Fig. 1 of the main text, is a maximally-mixed state and then perform projective measurements $M_{a|x}$ in three (or two) mutually-unbiased bases: \hat{X} , \hat{Y} , and \hat{Z} (or \hat{X} and \hat{Z}). In Fig. 1(a), we plot the temporal-SW as a function of the evolution time t . We can see that the temporal-SW always remains the maximal value of unity if there is no decay, while the temporal-SW decreases monotonically when γ_1 is non-zero, as expected; the dynamics of this system is Markovian.

EXAMPLE 2: A SIMPLE NON-MARKOVIAN MODEL: A QUBIT COHERENTLY COUPLED TO ANOTHER QUBIT

Our second example is that of a qubit coherently-coupled to another qubit. If we treat one qubit as the system and the other one as the environment (by tracing it out), we have a

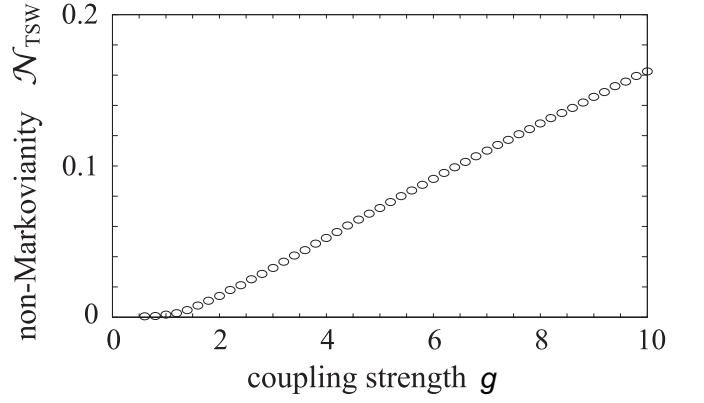


FIG. 2. (Color online) The degree of the non-Markovianity for a multimode reservoir with Lorentzian spectral density (example 3). The non-Markovianity N_{TSW} , defined by the temporal steerable weight, as a function of the coupling strength g . Here, g is in units of spectral width ω_w .

very simple example of a non-Markovian environment. The total Hamiltonian of the system in the interaction picture is

$$H_{\text{int}} = \hbar J (\sigma_+^1 \sigma_-^2 + \sigma_-^1 \sigma_+^2), \quad (11)$$

where σ_+^i and σ_-^i are the raising and lowering operators of the i th qubit, and $\hbar J$ is the coherent coupling between the system and the environment. We assume the system qubit is also subject to an intrinsic decay with decay rate γ_2 . In Fig. 1(b), we plot the temporal-SW for various decay rates γ_2 , after tracing out the effective environment-qubit. The initial condition of the system-qubit is that of a maximally-mixed state, while the environment-qubit is in its excited state. As seen in Fig. 1(b), there is a vanishing and a reappearance of the temporal-SW of the system qubit. Since we know that the temporal-SW should decrease monotonically under a Markovian dynamics, the oscillation of temporal-SW naturally shows that the qubit is undergoing non-Markovian evolution. This memory effect in this simple example is easy to understand in that information regarding the state of the system-qubit flows to the environment-qubit and returns at a later time; one cannot assume that the evolution of the environment is not influenced by its history.

EXAMPLE 3: A QUBIT COUPLED TO A NON-MARKOVIAN MULTIMODE RESERVOIR

In general, the dissipation γ rate in a Master equation description of an open-quantum system can be time-dependent, i.e. $\gamma = \gamma(t)$. If $\gamma(t) < 0$, it indicates that information can flow back to the system and the system dynamics can be non-Markovian. To show that the temporal-SW is sensitive to this, we use the same example as in Breuer *et al.* [5], where a qubit is coupled to a reservoir with a Lorentzian spectral density. In

this case, the decay rate can be written as

$$\gamma(t) = -\frac{2}{G(t)} \frac{d}{dt} |G(t)|, \quad (12)$$

where

$$G(t) = e^{-\omega_w t/2} \left[\cosh\left(\frac{bt}{2}\right) + \frac{\omega_w}{b} \sinh\left(\frac{bt}{2}\right) \right] \quad (13)$$

with $b = \sqrt{\omega_w^2 - 2g\omega_w}$. Here, g denotes the coupling strength and ω_w is the spectral width. We choose a mixed state as the initial state and plot the non-Markovianity \mathcal{N}_{TSW} as a function of g/ω_w in Fig. 2. Our results agree well with those in Ref. [6]: the non-Markovianity is zero when $g/\omega_w < 0.5$, and increases monotonically as a function of g/ω_w .

* yuehnan@mail.ncku.edu.tw

- [1] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, “Quantifying Einstein-Podolsky-Rosen steering,” *Phys. Rev. Lett.* **112**, 180404 (2014).
- [2] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming,” (2008).
- [3] M. Andersen, J. Dahl, and K. Vandenberghe, *CVXOPT: Python software for convex optimization* (2014). url:<http://cvxopt.org/>.
- [4] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, 2004).
- [5] H.-P. Breuer, E.-M. Laine, and J. Piilo, “Measure for the degree of non-Markovian behavior of quantum processes in open systems,” *Phys. Rev. Lett.* **103**, 210401 (2009).
- [6] E.-M. Laine, J. Piilo, and H.-P. Breuer, “Measure for the non-Markovianity of quantum processes,” *Phys. Rev. A* **81**, 062115 (2010).